Mean Reversion in Capital Market

White Paper

24/Dec/2008

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Abstract

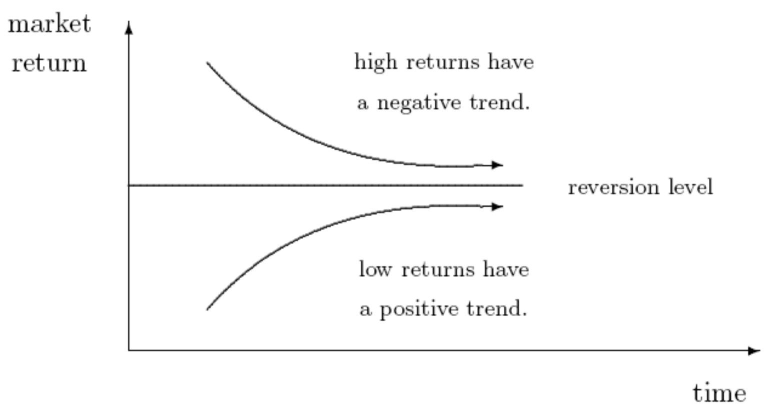
In trading, mean reversion processes are encountered, created and utilized to make trade winning strategies. As we know the stock market is volatile i.e. Price will not stay steady, price increases are followed by price declines, and vice versa. All prices will eventually move back towards the mean or average return i.e. Mean Reversion Level. This level helps us to take decision on purchasing or selling a stock. In this paper we focus on Concept of Mean Reversion, its estimation methods, the probability that in how many days stock price will revert back to the reversion level when we have a certain hypothetical reversion level. And also we will focus on the Vasicek model with stochastic volatility. We introduced stochastic volatility and ARCH, GARCH processes and gave some parameter estimation methods.

Key Words: Mean Reversion, Vasicek Model, Regression, Stochastic Volatility, ARCH, GARCH process

Section I: Mean Reversion

Introduction

Suppose the price power of a stock jumps to a certain level, it will certainly not stay there, we know that the price will soon revert to some reasonable level. This is known as mean reversion level. When the current market price is less than the average price (Mean Reversion Level), the stock is considered attractive for purchase, with the expectation that the price will rise. When the current market price is above the average price, the market price is expected to fall.



Identifying whether a market price series follow a mean reversion or not, we have certain tests as follows –

a) Augmented Dickey Fuller Test b) Phillips-Perron Test c) Variance Ratio Test

Mean Reversion Models

There are several mean reversion models used to estimate mean reversion level, reversion speed, etc.

Most of the models used are Vasicek Model, CIR Model, No-Arbitrage Models, Two-Factor Equilibrium Models, Ho-Lee Model and Hull-White Model which can be in the form of One Factor, Two Factor or Three Factor. Some of the models are explained as follows,

a. Vasicek One factor Model. In Vasicek's model, the risk neutral process for interest rate r is:

$$dr = a \cdot (b - r) \cdot dt + \sigma \cdot dz$$

Where a, b and σ are constants. This model incorporates mean reversion. The short rate is pulled to a level b at rate a. Superimposed upon this "pull" is a normally distributed stochastic term $\sigma \cdot dz$.

Limitations of One-factor models:

- Fail to capture the term structure of forward rates
- Fail to treat the long-run mean reversion as dynamic

Methodology

Following methods are used to estimate the parameters of mean reversion model.

- Regression Method
- Maximum Likelihood Method
- Method of Moments
- Nonparametric methods

The regression method is explained in next section.

Problem Description

We have the ratio price of two companies ONCG and CAIR. Both prices are correlated to each other. The primary objective is to find the mean reversion level of the price ratio.

And suppose the mean reversion level is 5 and the earlier mean reversion level is 7 then by what time it the level will come down form 7 to 5

To find probability of how many days the series crossing the +- 1sigma limits

We have the data for 2 years. Regressed the price ratio on its lag. Used regression estimates to calculate mean reversion level and mean reversion speed. These parameters further used to forecast.

The model is-

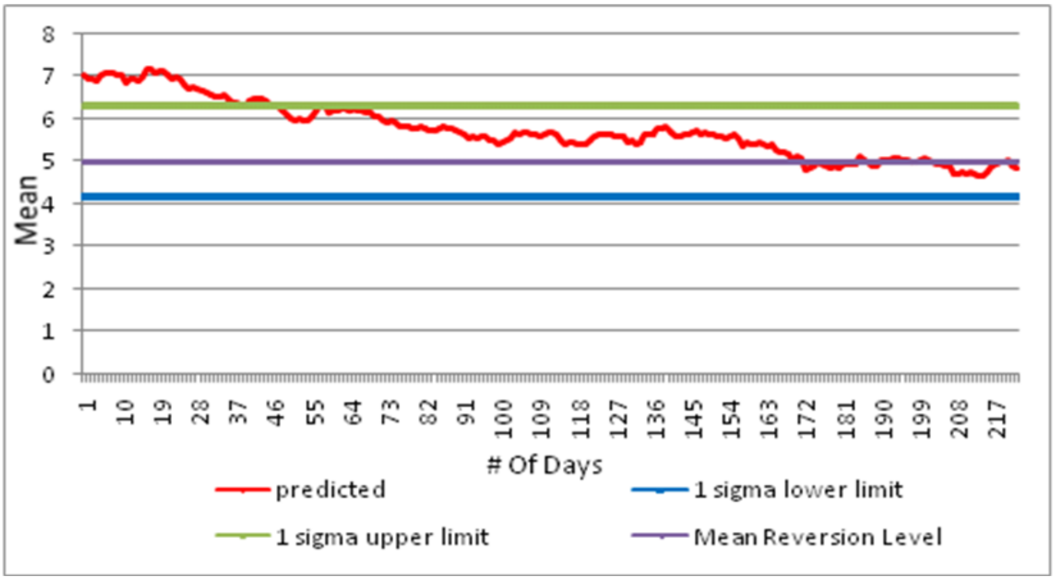
x (ti) = 0.06617 + 0.9867 x(ti-1) + 0.1604e(ti)

Estimated parameters are-

Mean Reversion level = 4.9752 (approx. 5)

Mean Reversion Speed = 0.0134

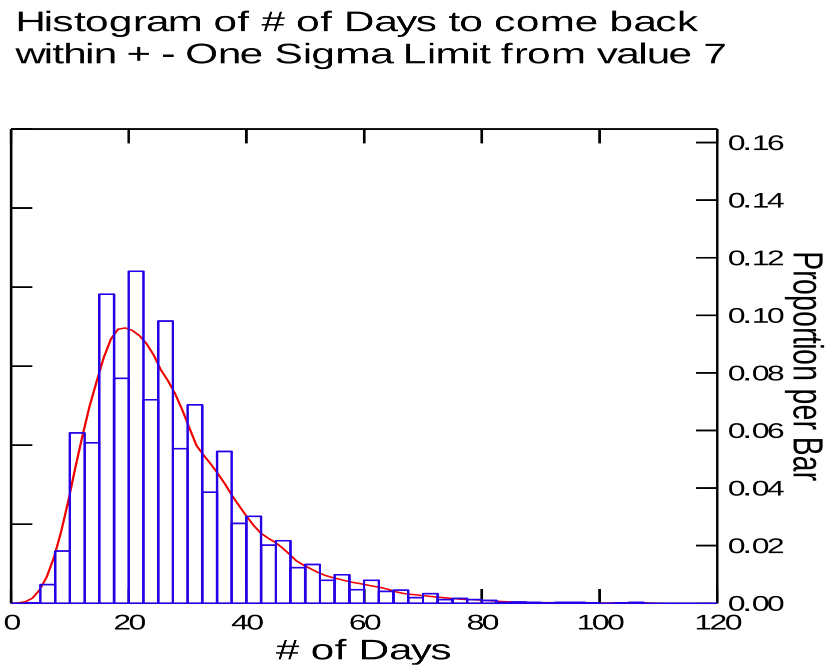
Mean Reversion Graph



We found the distribution of: after how many days, the ratio value comes back within +- One Sigma limit, which is roughly (4.1810415, 6.3189).

Suppose the starting ratio value is 7.

	# of Days
Arithmetic Mean	27
Standard Deviation	13.16
Percentiles	
50%	24
75%	33
90%	44
95%	53



From above graph we can interpret that on an average the price ratio 7 will come in one sigma limit within 53 days with 95% confidence.

Similar analysis is done by taking starting ratio as 8, 3 and 9. For results see appendix 3

Section II: Mean Reversion Model with Volatility

In the methodology mentioned above we have estimated the parameters with a constant volatility. In this section we will see how the mean reversion estimation will be after incorporating time dependent volatility.

Suppose that the short rate interest rate satisfies the mean-reverting Vasicek model with stochastic volatility where volatility also satisfies the mean-reverting Vasicek model, that is

$$dr_t = \varphi(b - r_t)dt + \sigma_t dW_{1t} \quad (1.1)$$

$$d\sigma_t = \gamma(k - \sigma_t)dt + \varepsilon dW_{2t} \quad (1.2)$$

where W1 and W2 are two independent Wiener processes and φ, b, γ and k are constants. Here ε is the volatility of the volatility process

Our aim is to estimate the parameters of the above model. In the application part since we will use discrete data, then we need to use a discrete model. So, first we need to discretisize this model. For discretization we will use Euler method.

Applying the Euler method to (1.1) and (1.2) we get

$$r_{t+1} - r_t = \varphi\,(b - r_t)\,\Delta t + \sigma_t(\Delta t)^{1/2}Z_{1t}$$

$$\sigma_{t+1} - \sigma_t = \gamma\,(k - \sigma_t)\,\Delta t + \varepsilon\,(\Delta t)^{1/2}Z_{2t}$$

where Z1t and Z2t are independent and identically distributed (i.i.d.) standard normal distributions, i.e. Z1,Z2 ~ N(0, 1), and ε is constant. If we take the observations daily, i.e. Δt = 1, and rearrange the equations we get

$$r_{t+1} = \varphi\, b + (1 - \varphi)r_t + \sigma_t Z_{1t}$$

$$\sigma_t = \gamma\, k + (1 - \gamma)\sigma_{t-1} + \varepsilon\, Z_{2t}$$

For simplicity, take $\alpha = \varphi\, b$, $\beta = (1 - \varphi)$, $a = (1 - \gamma)$, $x = \gamma\, k$ and $\xi t = \varepsilon\, Z_{2t}$ where

ξt is the innovation term. Then

$$r_{t+1} = \alpha + \beta r_t + \sigma_t Z_{1t} \quad (1.3)$$

$$\sigma_t = x + a\sigma_{t-1} + \xi_t \quad (1.4)$$

and we will work on these equations.

Parameter Estimation:

1. Tests:

Before starting the analysis we need to do some tests mentioned below

a) Ljung-Box Q-statistic:

This test is used to test whether there is autocorrelation or not.

H0: There is no autocorrelation. Vs. H1: There is autocorrelation.

$$Q_{LB} = (n(n+2)\sum_{j=1}^{h}\frac{\rho^2(j)}{n-j}$$

The hypothesis of autocorrelation is rejected if QLB > $\chi^2_{1-\alpha;h}$ where is the percent point function of the chi-square distribution

b) Arch Test:

Here the null is,

H0: There is no ARCH effect Vs. H1: There is ARCH effect

This test is to test whether there is any higher order ARCH effect or not

2. Methodology:

We are using Method of Moments to estimate the parameters of the model.

$$r_t = \alpha + \beta r_{t-1} + \sigma_{t-1} Z_{1t} \quad (1.5)$$

$$\sigma_{t-1} = x + a\sigma_{t-2} + \xi_{t-1} \quad (1.6)$$

We know that σ has the normal distribution with

Mean $M = x / 1\text{-}a$ and

Variance $\Sigma^2 = \varepsilon^2 / 1\text{-}a^2$

By ARCH / GARCH methods, we are estimating the volatility and by solving the simultaneous nonlinear system of equations we can estimate α, β, M and Σ parameters.

The nonlinear systems of equations are as follows-

$$
\begin{aligned}
&E[(r_t - \alpha - \beta r_{t-1})] = 0 \\
&E[(r_t - \alpha - \beta r_{t-1})^2] = M^2 + \Sigma^2 \\
&E[(r_t - \alpha - \beta r_{t-1})^3] = 0 \\
&E[(r_t - \alpha - \beta r_{t-1})^4] = 3(M^4 + 3\Sigma^4 + 6\Sigma^2 M^2)
\end{aligned}
$$

After solving these equations we can get the values of α, β, M and Σ, which we can use in equation (1.5) and (1.6)

References

1. A Research paper: ***A Stochastic Processes Toolkit for Risk Management*** by Damiano Brigo, Antonio Dalessandro, Matthias Neugebauer, Fares Triki (15 November 2007)

2. Google Book: Interest ***Rate Models - Theory and Practice*** by Damiano Brigo, Fabio Mercurio

3. http://www.puc-rio.br/marco.ind/revers.html (Mean Reversion models)

4. http://www.puc-rio.br/marco.ind/sim_stoc_proc.html (Simulation Technique)

5. Research paper: ***Stochastic Volatility, A New Approach For Vasicek Model With Stochastic Volatility*** by Serkan Zeytun (September 2005)

Section III – Case Study I

We also try to estimate the Mean Reversion Level and Mean Reversion Speed by taking log of the ratio.

We have the monthly average interest rate of US market from July 2001 to Nov 2008. The average of the rate is 2.52

(Source of Data: http://econstats.com/r/r_em1.htm)

By using first method as in case study1, we got following parameter estimates –

Variable	Parameter Estimate
Intercept	-0.0253
lag	0.9939

And the Mean Reversion parameters are

M= -4.1628

MS = 0.0061

Here we can see that the Mean Reversion Level is -4.1628 which seems to be irrelevant.

And by second method which is given in Appendix we got following parameters-

Variable	Parameter Estimate
Intercept	-0.1070
log_lag	0.0847

Here M = 3.5389 and

MS = 0.9219

Conclusion: From the above example it seems that a log transformation can be more appropriate than taking it as it is.

Appendix 2

Other method to estimate parameters of Mean Reversion by taking *log* of the price

Parameters Estimation:

Let P is the price of a stock and X=ln (P). Consider that x follows the arithmetic Ornstein-Uhlenbeck process toward an equilibrium level m

$$dx = \eta\,(m - x)\,dt + \sigma\,dz$$

(See Dixit & Pindyck, p.76; or Schwartz, 1997, footnote 15)

It is the limiting case (Δt tends to zero) of the AR (1) process:

$$x_t - x_{t-1} = m\,(1 - e^{-\eta\,\Delta t}) + (e^{-\eta\,\Delta t} - 1)\,x_{t-1} + \varepsilon_t$$

Where e t is normally distributed with mean zero and standard deviation se, and:

$$\sigma_\varepsilon^2 = [1 - \exp(-2\,\eta)]\,\sigma^2/2\eta$$

In order to estimate the parameters of mean-reversion, run the regression:

$$x_t - x_{t-1} = a + b\,x_{t-1} + \varepsilon_t$$

Calculate the parameters (it is easy to see why by looking the AR (1) equation):

$$m = -a/b$$

$$\eta = -\ln(1 + b)$$

$$\sigma = \sigma_\varepsilon \sqrt{\frac{2\ln(1+b)}{(1+b)^2 - 1}}$$

Where σ_ε is the standard deviation from the regression (the last equation corrects a small typo in Dixit & Pindyck).

If we use monthly data and we want to obtain annual values for the parameters, multiply the value of h, obtained in the equation above, by 12; and multiply the value of s obtained above by the square-root of 12, etc.

One important distinction between random walk and stationary AR(1) processes: for the last one all the shocks are transitory, whereas for random walk all shocks are permanent.

By this method we are getting,

M = 4.8708 and MS = 1.0125

Comparison of Mean Reversion level and Mean Reversion speed for above two methods-

	Method 1	Method 2
Mean Reversion Level	a/(1-b)	-a/b
Mean Reversion Speed	-ln(b)	-ln(1+b)

Simulation:

We have used Monte Carlo simulation to identify the probability that in how many days the price series will come in +- one sigma limit. We have followed following steps-

Initially consider the following Arithmetic Ornstein-Uhlenbeck process for a stochastic variable x (t):

$$\mathbf{dx = \eta\,(\bar{x} - x)\,dt + \sigma\,dz}$$

This means that there is a reversion force over the variable x pulling towards an equilibrium level   like a spring force.

In order to perform the simulation is necessary to get the discrete-time equation for this process. The correct discrete-time format for the continuous-time process of mean-reversion is the stationary first-order autoregressive process, AR (1), (for Δt = 1; the equation below is more general). So the sample path simulation equation for x (t) is performed by using the exact (valid for large Δt) discrete-time expression:

$$x_t = x_{t-1}\, e^{-\eta \Delta t} + \bar{x}\;(1 - e^{-\eta \Delta t}) + \sigma \sqrt{(1 - \exp(-2\,\eta\,\Delta t))/(2\,\eta)}\;\; N(0,1)$$

Suppose P is the price of the stock. And X=ln (P). Then the simulation equation will be-

$$P(t) = \exp\left\{ \left[\ln[P(t-1)]\, \exp[-\eta\,\Delta t]\right] + \left[\left[\ln(\bar{P}) - \frac{(\mu - r)}{\eta}\right] \left(1 - \exp[-\eta\,\Delta t]\right) \right] - \right.$$
$$\left. - \left[(1 - \exp[-2\,\eta\,t])\, \frac{\sigma^2}{4\,\eta} \right] + \sigma \sqrt{\frac{1 - \exp[-2\,\eta\,\Delta t]}{2\,\eta}}\;\; N(0,1) \right\}$$

(Please see http://www.puc-rio.br/marco.ind/sim_stoc_proc.html)

Using this simulation process simulate the x (t) (price) assuming hypothetical level as 7.

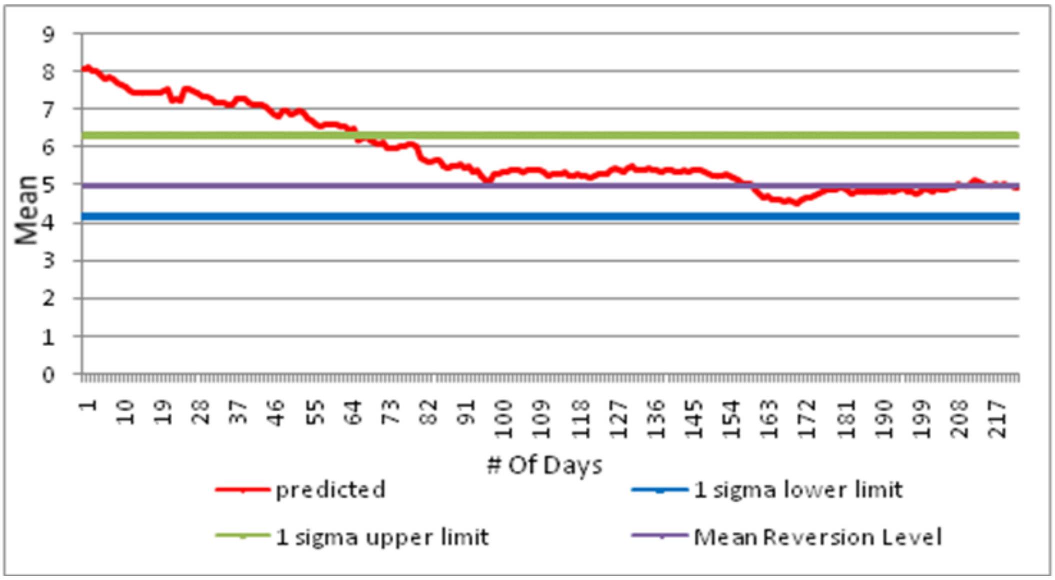
Calculate the +-one sigma limits for the price.

Count the number of days when the price will come back into +-one sigma limit and also calculate the percentile values for # of days.

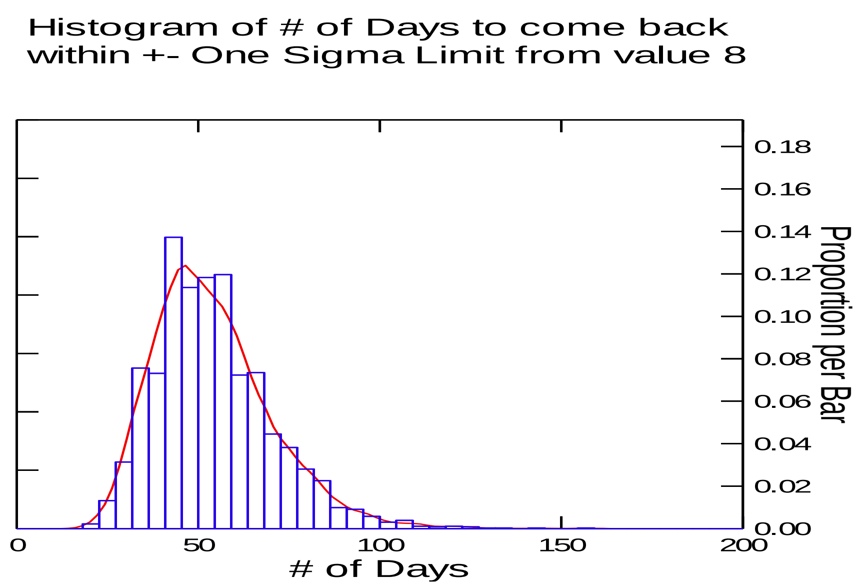
Appendix 3

If suppose the starting ratio value is 8.

Mean Reversion Graph



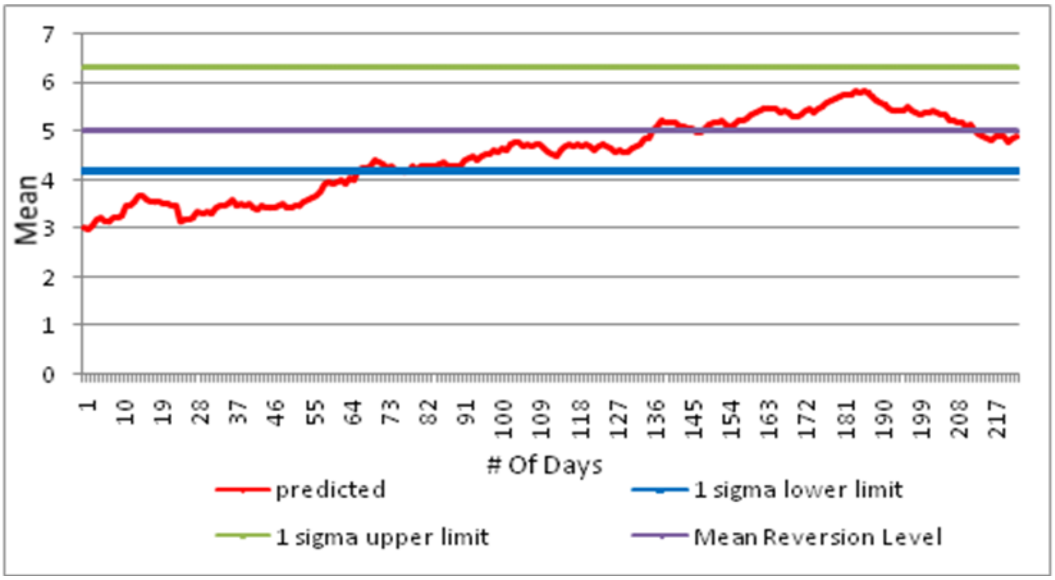
	# of Days
Arithmetic Mean	54
Standard Deviation	16.20
Percentiles	
50%	52
75%	63
90%	76
95%	84



From above graph we can interpret that on an average the price ratio 8 will come in one sigma limit within 84 days with 95% confidence

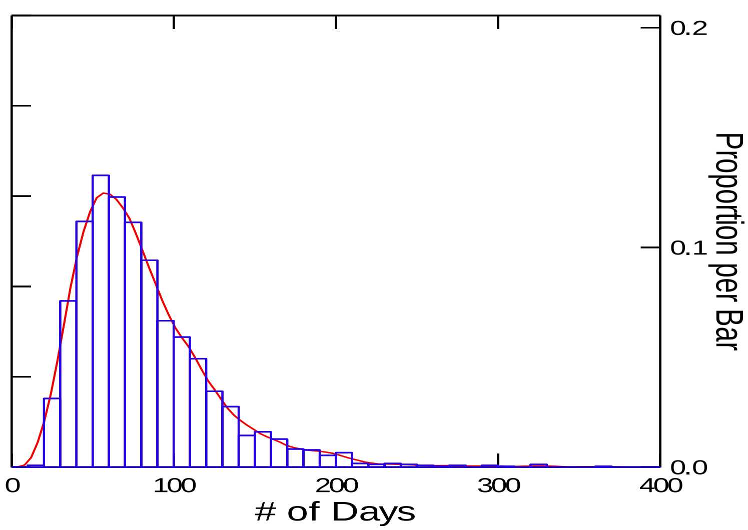
If suppose the starting ratio value is 3.

Mean Reversion Graph



	# of Days
Arithmetic Mean	81
Standard Deviation	41.25
Percentiles	
50%	72
75%	100
90%	133
95%	161

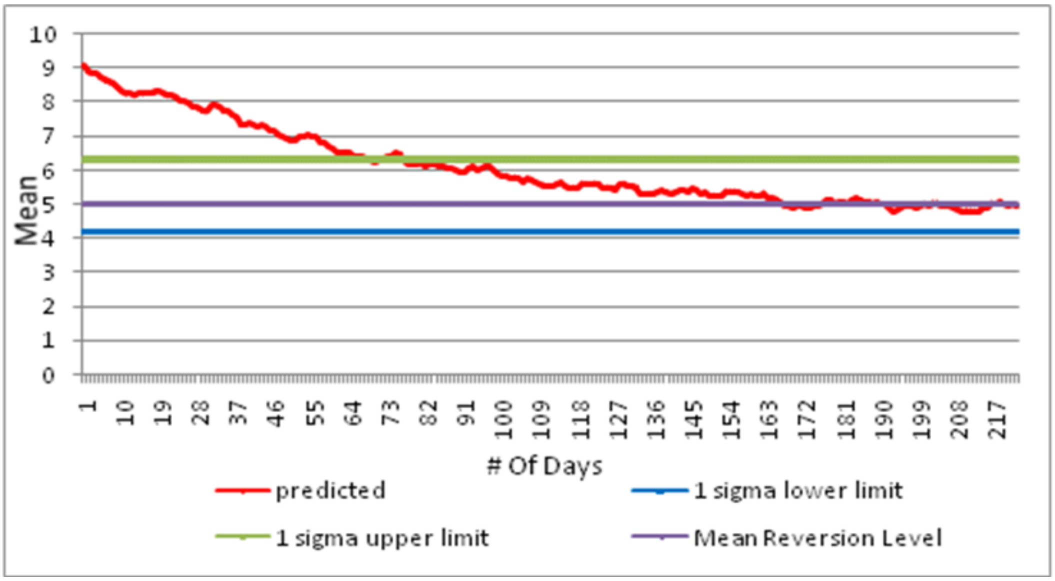
Histogram of # of Days to come back within +- One Sigma Limit from value 3



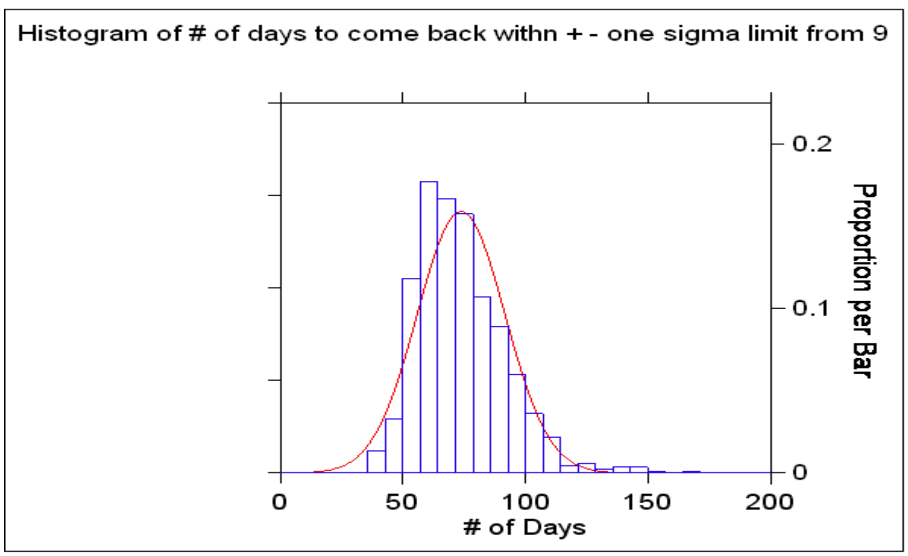
From above graph we can interpret that on an average the price ratio 3 will come in one sigma limit within 161 days with 95% confidence

If suppose the starting ratio value is 9.

Mean Reversion Graph



	# of Days
Arithmetic Mean	74
Standard Deviation	17.92
Percentiles	
50%	71
75%	84
90%	97
95%	106



From above graph we can interpret that on an average the price ratio 9 will come in one sigma limit within 106 days with 95% confidence